This exam has **two** problems:

- Problem 1 has four parts, and is worth 20 points.
- Problem 2 has five parts, and is worth 20 points. There is a bonus part in problem 2 which is worth 5 points.

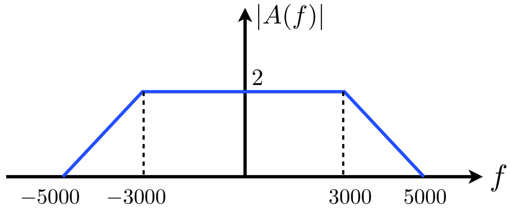
The Q-function is defined as

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

You can find the function evaluation for several values of x in the table below:

x	$Q(x)$
0	0.5000
0.1	0.4602
0.2	0.4207
0.3	0.3821
0.4	0.3446
0.5	0.3085
0.6	0.2743
0.7	0.2420
0.8	0.2119
0.9	0.1841
1.0	0.1587
1.1	0.1357
1.2	0.1151
1.3	0.0968
1.4	0.0808
1.5	0.0668
1.6	0.0548
1.7	0.0446
1.8	0.0359
1.9	0.0287
2.0	0.0228

Problem 1) Suppose a signal $A(t)$ is going to be transmitted over a channel. The magnitude of the Fourier transform of the $A(t)$ is given by



The maximum magnitude of $A(t)$ is 150: $|A(t)| \leq 150$. We use pulse coded modulation (PCM) to convert the continuous signal into a sequence of bits.

(a) What is the minimum sampling frequency required for perfect signal recovery? *[3 points]*

(b) We would like to design a uniform quantizer such that the average quantization noise power does not exceed 0.03. What is the maximum quantization interval width?

[8 points]

(c) What is the minimum number of quantization levels to satisfy the interval width requirement of part (b)? Assuming that the resulting sequence of bits is transmitted using BPSK, find the corresponding number of binary pulses per sample. *[5 points]*

(d) Let $x[n]$ be the binary sequence to be transmitted over the channel. To this end, first generate a baseband signal

$$x_b(t) = \sum_n x[n]\text{sinc}(t/T - n),$$

and then modulate $x_b(t)$ using a carrier frequency f_c, that is multiplying it by $\cos(2\pi f_c t)$, and send it over a wireless channel. Determine the maximum value for T and the minimum value for f_c. *[4 points]*

Problem 2) Consider a Gaussian communication channel

$$Y(t) = X(t) + Z(t)$$

where $X(t) \in \{-1, +1\}$ and $Z(t)$ is an additive white Gaussian noise distributed as $Z(t) \sim \mathcal{N}(0, \sigma^2)$.

(a) What is the signal to noise ratio of this channel? *[2 points]*

Upon observing Y, the receiver detects the input signal using the following rule:

$$\hat{X} = \begin{cases} +1 & \text{if } Y \geq 0 \\ -1 & \text{if } Y < 0. \end{cases}$$

(b) Find the bit error rate of this channel, i.e., $\Pr(\hat{X} \neq X)$. *[4 points]*

(c) A source generates integer numbers in $\mathcal{S} = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Transmitter firsts maps the source to a binary sequence using the table below,

s	0	1	2	3	4	5	6	7
$b_1b_2b_3$	000	001	010	011	100	101	110	111

Table 1: The binary representation map.

and sends the sequence by transmitting one bit at a time over the channel, using $0 \mapsto -1$ and $1 \mapsto 1$. At receiver, we first decode the bits, and remap the binary sequence to $\hat{s} \in \mathcal{S}$. Find the probability that $\hat{s} = s$. *[3 points]*

(d) Find the average mean squared distortion of this communication channel, that is $\mathbb{E}[|s - \hat{s}|^2]$, when $s = 4$ is sent and $\sigma = 2/3$. *[6 points]*

(e) Assume we want to use a better map (compared to Table 1) to reduce the average mean squared error. What is a desired property of a good map? *[5 points]*

(f) Give a good mapping which satisfies the criteria suggested in part (e). *[Bonus: 5 points]*